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Charged particle with magnetic moment in the Aharonov–Bohm potential

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Abstract. In this paper we will consider a charged quantum mechanical particle with a spin-$\frac{1}{2}$ and a gyromagnetic ratio of $g \neq 2$ in the field of a magnetic string. The interaction of the charge with the string is the well known Aharonov–Bohm effect, and the contribution of the magnetic moment associated with the spin in the case $g = 2$ yields additional scattering and zero modes (one for each flux quantum). The anomaly of the magnetic moment (i.e. $g > 2$) leads to bound states. We considered two methods for treating the case $g > 2$.

The first is the method of self-adjoint extension of the corresponding Hamilton operator. It yields one bound state as well as additional scattering. In the second we will consider three exactly solvable models for finite flux tubes and than consider the limit of zero radius. For a finite radius, there are $N + 1$ bound states (N is the number of flux quanta in the tube).

For $R \to 0$ the bound state energies tend to be infinite so that this limit is not physical. A sensible limit can be obtained by tending $g \to 2$ simultaneously with $R \to 0$. Thereby only for fluxes less than unity, the results of the method of self-adjoint extension are reproduced whereas for larger fluxes N bound states will exist. We conclude therefore that this method is not applicable.

We will discuss the physically interesting case of a small but finite radius whereby the natural scale is given by the anomaly of the magnetic moment of the electron $a_e = \frac{1}{2}(g - 2) \approx 10^{-3}$.

1. Introduction

There is a continuous interest in the study of scattering and bound states in the potential of a magnetic string

$$A = \frac{\Phi}{2\pi r} e_\varphi \tag{1}$$

with the flux Φ. The interaction of a charged particle with this potential is described by the minimal coupling

$$p \to p - \frac{e}{c} A \,. \tag{2}$$

The corresponding magnetic field

$$\mathcal{H} = \Phi \delta(x) \delta(y) e_3 \tag{3}$$

vanishes everywhere except on the flux line where it is infinite.

The famous Aharonov–Bohm effect [1] consists in non-trivial scattering of a charged particle off potential (1). It is due to the interference of phaseshifts of the wavefunction

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which are influenced by the potential (1). In an ideal situation the wavefunction vanishes where the magnetic field is non-zero, demonstrating the role of the potential. In the last few years, the AB effect was studied in connection with fractional spin and statistics in [3] and with its contribution to cosmic strings in [4, 5]. There is a close relation to the calculation of propagators in chromomagnetic background fields [6, 7].

While the initial investigation concerns a scalar particle, the inclusion of the spin is natural. In the case of a particle with spin s, there is an additional, with respect to (2), interaction of its magnetic moment

$$\mu = g\mu_B s/\hbar \tag{4}$$

(g being its gyromagnetic ratio) with the magnetic field (3), contributing

$$\Delta\hat{H} = \mu\mathcal{H} \tag{5}$$

to the Hamiltonian. As it stands, this is a point-like interaction and must be treated in an appropriate manner (see e.g [8, 9]). From the mathematical point of view, one has to consider the corresponding Hamilton operator on a domain of functions vanishing on the flux line so that the term with the δ-function disappears. On this domain, the operator is not self-adjoint and its self-adjoint extensions (a one-parameter family labelled by λ) define all possible point interactions (3).

In the case of a neutral particle with magnetic moment (i.e. with interaction (5)), this can be found in book [9] within a general mathematical framework. For a spinor particle, using the Dirac equation, this analysis has been done in [4, 10, 14].These authors have shown that the self-adjoint extensions can be defined by proper boundary conditions on the wavefunction on the flux line. Also, an analysis using a regularized δ-function was done in [4] and [16]. Therein was the possibility of a bound state discussed above. Similar results exist for spin-1 case [12].

In general, the Dirac equation leads to a magnetic moment which is characterized by a gyromagnetic ratio of $g = 2$. This case is exceptional from the point of view of its interaction with a magnetic flux line because the repulsive force of the AB effect is exactly compensated for by the attractive force of the interaction of the magnetic moment with the flux (in the case when they are antiparallel). This produces zero modes, i.e. bound states of zero binding energy. This situation was probably first mentioned in [2] where it was shown that a spin-$\frac{1}{2}$ particle† in a magnetic field (in general, non-singular) of total flux $\Phi/(hc/e) = N + \tilde{\delta}$, $0 < \tilde{\delta} < 1$ has N zero-energy normalizable eigenstates. It has the remarkable property that its Hamilton operator factorizes, and both equations have essentially the same form. This is an example of a supersymmetric quantum mechanical system.

Now it is clear that an anomalous magnetic moment destroys this property. Having in mind realistic particles such as the electron with its anomaly factor

$$a_e \equiv \frac{g-2}{2} = 0.001\,159 \tag{6}$$

we are considering in the present paper a particle with spin s and the gyromagnetic ratio g in 3 different, exactly solvable models of regularization of the δ-function by a flux tube of radius R. We are also establishing their connection with the approach of self-adjoint extensions. We should consider to what extend these models correspond to different extensions. In each model there are $N+1$ bound states, if the gyromagnetic ratio g is larger than two and the magnetic moment is directed anti-parallel to the magnetic flux, otherwise

† Described by the Pauli (with $g = 2$) and also by the Dirac equation.

there are no bound states. This is one bound state more than there are zero modes in the case of $g = 2$. When shrinking the radius of the flux tube to zero, the gyromagnetic ratio must tend to 2 in order to have finite bound-state energy.

The models we use are ($\mathcal{H} = H(r)e_3$)

$$\text{(i)}\quad H(r) = \frac{\Phi}{\pi R^2}\Theta(R-r) \qquad \text{(homogeneous magnetic field inside)} \tag{7}$$

$$\text{(ii)}\quad H(r) = \frac{\Phi}{2\pi r R}\Theta(R-r) \qquad \text{(magnetic field proportional to } 1/r \text{ inside)} \tag{8}$$

$$\text{(iii)}\quad H(r) = \frac{\Phi}{2\pi R}\delta(r-R) \qquad \text{(a cylindrical shell with } \delta\text{-function)}\,. \tag{9}$$

For simplicity, we consider a non-relativistic Hamilton operator

$$\hat{H} = \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 + \mu\mathcal{H} \tag{10}$$

with μ given by (4).

Due to spin conservation, the magnetic interaction (5) can be replaced by

$$\pm\tfrac{1}{2}g\mu_{\mathrm{B}}H(r) \tag{11}$$

$\pm$ corresponding to the spin projection on the flux line. In what follows, we restrict ourselves to the minus sign, in which the magnetic moment leads to a binding force. Furthermore, we choose $\Phi > 0$; for $\Phi < 0$ the spin direction must be reversed.

This paper is organized as follows. In the next section we consider the point interaction as self-adjoint extension and obtain boundary conditions on the wavefunction at the origin. In the following section, we consider magnetic flux tubes (7)–(9) with finite radius R and write down the wavefunctions for bound states and scattering states. In the fourth section, we consider the limit $R \to 0$ and establish its connection with the self-adjoint extension and look into the physical consequences. Conclusions are given in the last section.

2. Self-adjoint extension

The Schrödinger equation for the problem considered here can be written in the form

$$\left(\frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 - \frac{g\mu_{\mathrm{B}}H(r)}{2}\right)\psi = E\psi \tag{12}$$

where A and H are given by (1) and (3) for an infinitely thin flux tube and by (7)–(9) for a finite flux tube, respectively. After the separation of the angular dependence and the translational motion parallel to the flux tube by

$$\psi(x) = \sum_{m=-\infty}^{\infty}\psi_m(r)\frac{\mathrm{e}^{-im\varphi}}{\sqrt{2\pi}}\frac{\mathrm{e}^{ip_3x_3}}{\sqrt{2\pi}} \tag{13}$$

the equation reads (for simplicity, we set $p_3 = 0$)

$$\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{(m-\delta a(r))^2}{r^2} - \frac{g\delta h(r)}{2}\right)\psi_m(r) = \epsilon\,\psi_m(r) \tag{14}$$

with

$$A = \frac{\phi}{2\pi r}a(r)\,.$$

$\delta = \Phi/(hc/e)$ is the flux measured in units from the flux quantum, $h(r) = \frac{1}{r}\frac{\partial}{\partial r}a(r)$ is the radial distribution of the magnetic field (it is normalized according to $\int_0^\infty \mathrm{d}r\, r h(r) = 1$) and with the energy $\epsilon = E/(2m/\hbar^2)$.

We shall consider the case of an infinitely thin flux tube: $a(r) = 1$ and $h(r) = (1/r)\delta(r)$. Being inserted into the Schrödinger equation (14), this is a potential with concentrated support. The standard technique to handle them is the method of self-adjoint extension (see the textbook [9], where it is extensively used in quantum mechanics). One starts with the Hamilton operator

$$\hat{H}_0 = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{(m-\delta)^2}{r^2} \tag{15}$$

which coincides with the Hamilton operator $\hat{H}$ of equation (14) in the interval $r \in (0, \infty)$, i.e. everywhere except at the point $r = 0$. This is equivalent to considering $\hat{H}$ on the domain of functions

$$\mathcal{D}(H_0) = \left\{\psi \in \mathcal{L}^2([0,\infty)) \mid \psi(0) = 0\right\} \tag{16}$$

i.e. on functions vanishing as $r \to 0$. After that one observes than the operator $\hat{H}_0$ on $\mathcal{D}$ is symmetric but not self-adjoint. By enlarging the domain $\mathcal{D}$ in a suitable manner to $\tilde{\mathcal{D}}$, $\hat{H}$ can be extended to become self-adjoint on $\tilde{\mathcal{D}}$. This self-adjoint extension can than be considered as a correct definition of the initial operator $\hat{H}$ in equation (14) involving the δ-function potential.

In our case this procedure is as follows. The eigenfunctions of $\hat{H}_0$ are

$$\psi_m(r) = J_{|m-\delta|}\left(\sqrt{\epsilon}r\right) \tag{17}$$

($m = 0, \pm 1, \dots$). The scalar product

$$(\varphi, H_0\psi) = \int_0^\infty \mathrm{d}r\, r\varphi(r)\hat{H}_0\psi(r) \tag{18}$$

implies that $\hat{H}_0$ is symmetric if

$$\lim_{r\to 0}\left(\varphi^* r\frac{\partial}{\partial r}\psi - r\frac{\partial}{\partial r}\varphi^*\psi\right) = 0\,. \tag{19}$$

For the functions $\psi_m(r)$ (17), this is clearly fulfilled.

Now, N shall be the integer part of the flux

$$\delta = N + \tilde{\delta} \quad (0 \leqslant \tilde{\delta} < 1)\,.$$

For any $\psi \in \mathcal{D}$, there are functions $\varphi \notin \mathcal{D}$, fulfilling condition (19). Their behaviour as $r \to 0$ is

$$\varphi \sim r^{-\tilde{\delta}} + \lambda r^{\tilde{\delta}} \qquad \lambda \text{ is real}\,. \tag{20}$$

The corresponding eigenfunctions are given below. Adding them to the domain $\mathcal{D}$ (16) of the operator $\hat{H}_0$ we obtain the enlarged domain $\tilde{\mathcal{D}}$. It can be shown that $\hat{H}_0$ on $\tilde{\mathcal{D}}$ is self-adjoint. The parameter λ is arbitrary. Its dimension is $r^{-2\delta}$. By that different choices of λ lead to different self-adjoint extensions.

The corresponding eigenfunctions in the continuous part of the spectrum ($\epsilon = k^2$) are

$$\psi_S(r) = J_{\tilde{\delta}}(kr) + B_N(k)\, H^{(1)}_{\tilde{\delta}}(kr)\,. \tag{21}$$

In general, the scattering amplitude is defined by the asymptotics of the wavefunction as $r \to \infty$:

$$\psi(x) \approx \mathrm{e}^{\mathrm{i}kr\cos\varphi} + f(k,\varphi)\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} \tag{22}$$

where φ is the scattering angle. The usual Aharonov–Bohm scattering (in other words, scattering without magnetic moment) corresponds to the first term in the RHS of (21), and the scattering amplitude is well known. In expanding

$$f(k,\varphi) = \sum_{m=-\infty}^{\infty} f_m(k)\frac{e^{-im\varphi}}{\sqrt{2\pi}}$$

its contribution reads

$$f_m^{AB}(k,\varphi) = \frac{1}{\sqrt{k}}\left(e^{i\pi(m-|m+\delta|)} - 1\right) e^{-i\pi/4}. \tag{23}$$

In that respect, the presence of the contribution of the Hankel function in RHS of (21), which describes a outgoing cylindrical wave, leads to an additional contribution to the scattering amplitude

$$f_m(k) = f_m^{AB}(k) + \delta_{m,N}\frac{1}{\pi\sqrt{k}}B_N(k). \tag{24}$$

There is one eigenfunction describing a bound state with binding energy $\kappa = -\epsilon$:

$$\psi_B(r) = K_{\tilde{\delta}}\left(\sqrt{\kappa}r\right). \tag{25}$$

Now, by expanding solutions (25) and (21) as $r \to 0$, we can obtain the connection of the bound state energy κ with the parameter λ of the self-adjoint extension

$$\lambda = -\frac{\Gamma(1-\tilde{\delta})}{\Gamma(1+\tilde{\delta})}\left(\sqrt{\kappa}/2\right)^{2\tilde{\delta}} \tag{26}$$

by using (20). From this formula it follows that the bound state would occur only in the case of a negative parameter of the extension λ. For the scattering states, we obtain from (22) and (20)

$$B_N(k) = \frac{i\sin\pi\tilde{\delta}}{e^{-i\pi\tilde{\delta}} + \frac{\Gamma(1+\tilde{\delta})}{\Gamma(1-\tilde{\delta})}\lambda(k/2)^{2\tilde{\delta}}}. \tag{27}$$

Hence, for any parameter λ of the extension, there is an additional scattering, and for $\lambda < 0$ there is a bound state. In the latter case, the scattering amplitude can be expressed in terms of the bound state energy

$$B_N(k) = \frac{i\sin\pi\tilde{\delta}}{e^{-i\pi\tilde{\delta}} - \left(\kappa/k^2\right)^{\tilde{\delta}}}. \tag{28}$$

3. Three models

The regularization of the δ-function interaction can be done by many different models for a finite flux tube. We consider here the simplest examples that are exactly solvable. We write down the wavefunctions inside the tube and connect them with the outside function.

The outside function ($r > R$) is an eigenfunction of the Hamilton operator (15). In the case of $\epsilon < 0$, this eigenfunction is given by

$$\psi_m(r) = K_{m-\delta}(\sqrt{-\epsilon}r) \tag{29}$$

and describes the bound state solution. Its logarithmic derivative is

$$R_{ex} \equiv R\frac{\partial}{\partial r}\ln\psi_m(r)_{|r=R+0} = \sqrt{-\epsilon}R\frac{K'_{m-\delta}\left(\sqrt{-\epsilon}R\right)}{K_{m-\delta}\left(\sqrt{-\epsilon}R\right)}. \tag{30}$$

In $\epsilon = k^2 > 0$, we obtain the outside scattering solution ($r > R$)

$$\psi_m(r) = J_{|m-\delta|}(kr) + B_m(k)\, H^{(1)}_{|m-\delta|}(kr) \tag{31}$$

and its logarithmic derivative is

$$R_{\mathrm{ex}} \equiv r\frac{\partial}{\partial r}\ln\psi_m(r)_{|_{r=R+0}} = \sqrt{\epsilon}R\frac{J'_{|m-\delta|}\left(\sqrt{\epsilon}R\right) + B_m(k)\,H^{(1)'}_{|m-\delta|}\left(\sqrt{\epsilon}R\right)}{J_{|m-\delta|}\left(\sqrt{\epsilon}R\right) + B_m(k)\,H^{(1)}_{|m-\delta|}\left(\sqrt{\epsilon}R\right)}\,. \tag{32}$$

Below we are interested in the limit $R \to 0$. For $\epsilon < 0$ we note that

$$R_{\mathrm{ex}} = \begin{cases} -|m-\delta| - 2|m-\delta|\dfrac{\Gamma(1-|m-\delta|)}{\Gamma(1+|m-\delta|)}\left(\dfrac{\sqrt{-\epsilon}R}{2}\right)^{2|m-\delta|} + \dots & |m-\delta| < 1 \\ -|m-\delta| - 2\dfrac{1}{|m-\delta|-1}\left(\dfrac{\sqrt{-\epsilon}R}{2}\right)^2 + \dots & |m-\delta| > 1 \end{cases} \tag{33}$$

where two cases have to be distinguished.

3.1. Homogeneous magnetic field

In this model the magnetic field is homogeneous inside and zero outside. The functions $h(r)$ and $a(r)$ are

$$h(r) = \frac{2}{R^2}\Theta(R-r), \quad a(r) = \frac{r^2}{R^2}\Theta(R-r) + \Theta(r-R)\,. \tag{34}$$

The Schrödinger equation is

$$\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{(m-\delta\frac{r^2}{R^2})^2}{r^2} - \frac{g}{2}\delta\frac{2}{R^2}\right)\psi_m(r) = \epsilon\psi_m(r)\,. \tag{35}$$

The solution which is regular in $r = 0$, is given by

$$\psi_m(r) = r^{|m|}\,{}_1F_1\left(\frac{2-g}{4} + \frac{|m|-m}{2} - \frac{\epsilon R^2}{4\delta}, 1+|m|, \delta\frac{r^2}{R^2}\right)\mathrm{e}^{-\delta r^2/(2R^2)}\,. \tag{36}$$

We need its logarithmic derivative in $r = R$

$$R_1 \equiv R\frac{\partial}{\partial r}\psi_m(r)_{|_{r=R-0}}$$

for $m \geqslant 0$, and we use the notation $x \equiv \sqrt{-\epsilon}R$:

$$R_1 = |m| - \delta + \delta\frac{\frac{2-g}{2} + \frac{x^2}{2\delta}}{1+|m|}\,\frac{{}_1F_1(\frac{2-g}{4} + 1 + \frac{x^2}{4\delta}, 2+|m|; \delta)}{{}_1F_1\left(\frac{2-g}{4} + \frac{x^2}{4\delta}, 1+|m|; \delta\right)}\,. \tag{37}$$

For $x \to 0$ we note that

$$R_1 = |m| - \delta + \frac{2-g}{2}\delta m\alpha_1 + x^2\beta_1 + \dots \tag{38}$$

with

$$\alpha_1 = \frac{1}{2(1+|m|)}\frac{{}_1F_1(\frac{2-g}{4}+1, 2+|m|; \delta)}{{}_1F_1(\frac{2-g}{4}, 1+|m|; \delta)} \qquad \beta_1 = \frac{\partial}{\partial g}\frac{2-g}{2}\alpha_1\,.$$

The properties $\alpha_1 > 0$ and $\beta_1 > 0$ can be checked.

3.2. *Magnetic field proportional to* $1/r$

In this model we have

$$h(r) = \frac{1}{rR}\Theta(R-r) \qquad a(r) = \frac{r}{R}\Theta(R-r) + \Theta(r-R)\,. \tag{39}$$

The corresponding equation is

$$\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\left(m-\delta\frac{r}{R}\right)^2}{r^2} - \frac{g}{2}\delta\frac{1}{rR}\right)\psi_m(r) = \epsilon\psi_m(r) \tag{40}$$

and it has a solution regular in $r=0$,

$$\psi_m(r) = r^{|m|}\,{}_1F_1\left(m\left(\frac{1}{2}+|m|-m\frac{\delta}{\bar\delta}\right)-\frac{g}{4}\frac{\delta}{\bar\delta}, 1+2|m|; 2\bar\delta\frac{r}{R}\right)\mathrm{e}^{-\bar\delta r/R} \tag{41}$$

with the notation $\bar\delta \equiv \sqrt{\delta^2-\epsilon R^2}$. Its logarithmic derivative is

$$R_2 \equiv r\frac{\partial}{\partial r}\psi_m(r)_{|r=R-0} = |m| - \bar\delta$$

$$+2\frac{(\frac{1}{2}+|m|)\bar\delta-(m+g/4)\delta}{1+2|m|}\frac{{}_1F_1\left(\frac{3}{2}+|m|-(m+g/4)\frac{\delta}{\bar\delta}, 2+2|m|; 2\bar\delta\right)}{{}_1F_1\left(\frac{1}{2}+|m|-(m+g/4)\frac{\delta}{\bar\delta}, 1+2m; 2\bar\delta\right)}\,.$$

For $m \geqslant 0$ and $x \equiv \sqrt{-\epsilon}R \to 0$ we note that

$$R_2 = m - \delta + \frac{2-g}{2}\delta\alpha_2 + \beta_2 x^2 + \ldots \tag{42}$$

with

$$\alpha_2 = \frac{1}{1+2m}\frac{{}_1F_1\left(\frac{2-g}{4}+1, 2+2m; 2\delta\right)}{{}_1F_1\left(\frac{2-g}{4}, 1+m; 2\delta\right)}$$

and

$$\beta_2 = \frac{1}{2\delta}\left((1+2m)\alpha_2 - 1 + \frac{2-g}{2}\left(\frac{\partial}{\partial\delta} - (g+4m)\frac{\partial}{\partial g}\right)\alpha_2\right).$$

Also in this case the properties $\alpha_2 > 0$ and $\beta_2 > 0$ can be checked.

3.3. *Cylindrical shell with* δ*-function*

Moving the δ-function from $r=0$ to $r=R$, one obtains a cylindrical shell on which the magnetic field is infinite:†

$$a(r) = \Theta(R-r) \qquad h(r) = \frac{1}{R}\delta(r-R)\,.$$

The radial equation reads

$$\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{(m-\delta\,\Theta(R-r))^2}{r^2} - \frac{g}{2}\delta\frac{1}{R}\delta(r-R)\right)\psi_m(r) = \epsilon\psi_m(r)\,. \tag{43}$$

In this case the δ-function, due to being moved away from the origin, can be treated as usual in one-dimensional case and substituted by the known, boundary conditions

$$r\partial_r\psi_m(r)\Big|_{R-0}^{R+0} = -\frac{g}{2}\delta\psi_m(r)_{|R}\,. \tag{44}$$

† This model is intensively used in [10].

Then the solution of equation (43) are Bessel functions

$$\psi_m(r) = \begin{cases} \alpha J_{|m|}(\sqrt{\epsilon} r) & \text{for } r < R \\ J_{|m-\delta|}(kr) + B_m(k)\, H^{(1)}_{|m-\delta|}(kr) & \text{for } r > R \end{cases} \tag{45}$$

with some coefficient α, and from condition (44) it follows that

$$\begin{aligned} R_3 &\equiv -\tfrac{1}{2} g\,\delta + r\tfrac{\partial}{\partial r}\psi_m(r)_{|r=R-0} \\ &= -\tfrac{1}{2} g\,\delta + \sqrt{\epsilon} R \tfrac{J'_{|m|}(\sqrt{\epsilon}R)}{J_{|m|}(\sqrt{\epsilon}R)} \qquad \text{for} \quad \epsilon > 0 \\ &= -\tfrac{1}{2} g\,\delta + \sqrt{-\epsilon} R \tfrac{I'_{|m|}(\sqrt{-\epsilon}R)}{I_{|m|}(\sqrt{-\epsilon}R)} \qquad \text{for} \quad \epsilon < 0 . \end{aligned}$$

For $x \equiv \sqrt{-\epsilon} R \to 0$ we have

$$R_3 = |m| - \delta + \frac{2-g}{2}\delta\alpha_3 + \beta_3 x^2 + \ldots \tag{46}$$

with

$$\alpha_3 = 1 \qquad \beta_3 = \frac{1}{2(1+|m|)} .$$

3.4. Bound state energy and scattering amplitude

The solutions in all three models are determined by the condition that

$$R_{\rm ex} = R_i \qquad (i = 1, 2, 3) . \tag{47}$$

There are scattering solutions for all values of the parameters. They can be obtained by solving (47). The scattering amplitude is

$$B_m(k) = -\left.\frac{\left(x\frac{\partial}{\partial x} - R_i\right) J_{|m-\delta|}(x)}{\left(x\frac{\partial}{\partial x} - R_i\right) H^{(1)}_{|m-\delta|}(x)}\right|_{x=kR} . \tag{48}$$

We shall now consider the bound-state solutions. They do not exist for all values of the parameters. In considering the behaviour of R_i and $R_{\rm ex}$ as functions of $x \equiv \sqrt{-\epsilon} R$, it can be seen that $R_{\rm ex}$ decreases starting from $R_{\rm ex}(0) = -|m-\delta|$ (cf (33)), while $R_i(x)$ increases starting from $R_i(x) = |m| - \delta + \frac{1}{2}(2-g)\delta m \alpha_i$ (cf (38), (42), (46). So, solutions with binding energy $\kappa_m \equiv \sqrt{-\epsilon} = x/R$ of equation (47) are possible for

$$g > 2 \qquad 0 \leqslant m < \delta\left(1 + \frac{g-2}{4}\alpha_i\right) . \tag{49}$$

In the case of $g = 2$, all solutions have vanishing energy; they correspond to zero modes. In general, the solution of (47) is

$$x = f(\delta, g, m) \qquad \text{with} \qquad x = \kappa_m R \tag{50}$$

where f is a dimensionless function. Some of the lowest solutions of equation (47) are shown in figure 1 for the model with the cylindrical δ-shell. Similar pictures can be drawn for the other two models. It can be seen that, in general, there is no simple rule for the energy levels $\kappa_m = x/R$ for general values of the parameters.

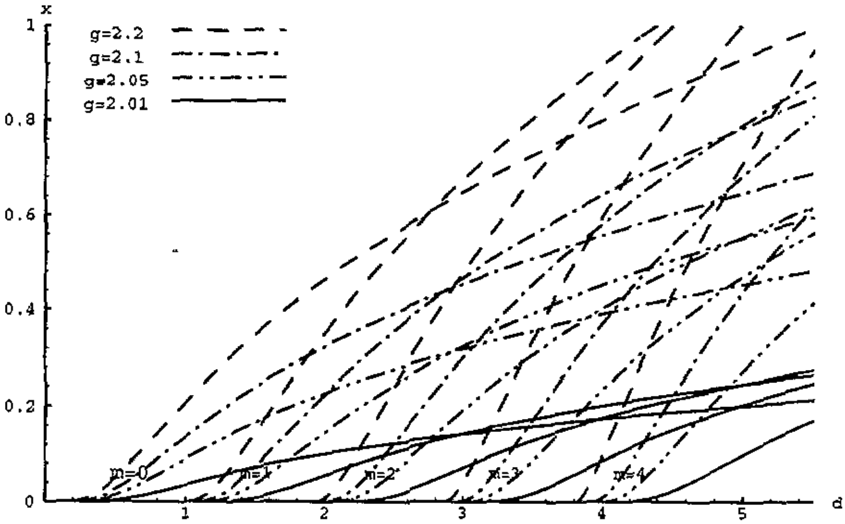

Figure 1. The solutions $x = f(d, g, m)$ of equation (50) for $m = 0, 1, \ldots 4$ and $g = 2.2, 2.1, 2.05, 2.01$.

4. The limit $R \to 0$

In the limit $R \to 0$, with all other parameters fixed, the bound state energy increases unbounded as can be seen from (50). This indicates that the limit $R \to 0$ in the models with the finite flux tube is not physical, at least not in the non-relativistic approximation chosen here.

One possibility of getting a sensible limit as $R \to 0$ is to let a_e go to zero together with R. Clearly, this is not meaningful, having in mind physical particles. But this makes possible both establishing the connection with the method of self-adjoint extension and, as will be seen below, clarifying its applicability to the system under consideration.

We will consider

$$a_e \equiv \frac{g-2}{2} \to 0.$$

In this case all solutions x of equation (50) tend towards zero, and the equation (47), which defines the bound states, can be approximated. We obtain for the highest angular momentum the equation

$$\frac{g-2}{2}\delta\alpha_i = \left(\frac{\sqrt{\kappa_N}R}{2}\right)^{2\tilde{\delta}} \frac{2\tilde{\delta}\Gamma(1-\tilde{\delta})}{\Gamma(1+\tilde{\delta})} \qquad (m = N) \tag{51}$$

where N is the integer part of the flux, and

$$\frac{g-2}{2}\delta\alpha_i = (\sqrt{\kappa_m}R)^2 \left(\beta_i + \frac{1}{4(|\delta - m| - 1)}\right) \qquad (m = 0, 1, \ldots, N-1) \tag{52}$$

for the lower angular momenta, using (33), (38), (42) and (46).

For $g \to 2$, R being fixed, κ_m $(m = 0, 1, \ldots, N-1)$ tend to zero in proportion to $g-2$, whereas κ_N behaves like $(g-2)^{1/\tilde{\delta}}$; i.e. tends more quickly to zero. Therefore, κ_m corresponds to the zero modes (for $g = 2$), whereas the state of κ_N has no equivalent in that case. It can be expected that its wavefunction vanishes.

In the case of $N = 0$, the flux for $R \to 0$ is less than unity and a finite binding energy of the only bound state can be obtained by substituting

$$\frac{g-2}{2} \to \frac{2R^{2\delta}}{\alpha_i}\frac{g-2}{2} \tag{53}$$

into the initial equation (12) and after that performing the limit $R \to 0$. After the substitution (53), the bound state energy is determined by

$$\left(\frac{\sqrt{\kappa_0}}{2}\right)^{2\delta} = \frac{\Gamma(1+\delta)}{\Gamma(1-\delta)}\frac{g-2}{2} \tag{54}$$

(instead of (51)), and we observe from equation (26) that the parameter λ of the extension is actually the anomaly of the magnetic moment (up to sign):

$$\lambda = -\frac{g-2}{2}. \tag{55}$$

This treatment of the δ-function is equivalent to the general approach to the two-dimensional δ-function in the Schrödinger equation by Berezin and Faddeev [8] and Albeverio [9], where the need to renormalize the coupling was pointed out.

In the case of fluxes larger than unity ($N \geqslant 1$), there are bound states with energy κ_m larger than κ_N; *renormalization (53) is not sufficient to keep these states finite.* Instead, using (52), one must substitute

$$\frac{g-2}{2} \to \frac{g-2}{2}R^2 \tag{56}$$

in the initial equation (12). Then, instead of (52), the binding energies are given by

$$\kappa_m = \frac{g-2}{2}\frac{\delta\alpha_i}{\beta_i + 1/(4(|\delta - m| - 1))} \tag{57}$$

($m = 0, 1, \ldots, N-1$). In this case we have $\kappa_N = 0$.

A different way of understanding the limit $R \to 0$ is to keep R small but finite. In that case there is a natural scale given, on the one hand, by the value of the anomaly of the magnetic moment of the electron

$$a_e = \frac{g-2}{2} = 0.001\,159$$

and on the other hand, by the bound state energy being non-relativistic, i.e. smaller than the electron mass because we are considering the non-relativistic Schrödinger equation.

In the case of a flux less than unity, the energy is non-relativistic for κ_0 much less than the inverse Compton wavelength of the electron, $\kappa_0 \ll 1/\lambda_c$, and the radius must satisfy the inequality

$$R \gg \left(\frac{g-2}{2}\frac{\Gamma(1+\delta)}{2\Gamma(1-\delta)}\alpha_i\right)^{1/(2\delta)} 2\lambda_c\,. \tag{58}$$

via equation (51). In that way, it can be considered to be smaller than λ_c so that the flux tube can be considered as thin. The considerations done here for small $x = \sqrt{\kappa}R$ should mean that the size of the orbit of the bound states is much larger than the radius R.

Similar considerations apply to the case of the flux being larger than unity. Here, the radius must satisfy

$$R \gg \left(\frac{g-2}{2}\delta\frac{1}{\beta_i + 1/4(|\delta - m| - 1)}\alpha_i\right)^{1/2} \lambda_c\,. \tag{59}$$

This condition is stronger than (58). Nevertheless, R may be made smaller than λ_c so that the flux tube can be thin in this case as well.

The limit $R \to 0$ for scattering states should be considered; i.e. $k^2 = \epsilon > 0$. Expanding $B_m(k)$ (48), the additional scattering amplitude for a given angular momentum m and energy k^2 for $x \to 0$ appears to be

$$B_m(k) = \frac{\mathrm{i}\sin\pi\nu\ (x/2)^{2\nu}}{\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)(2\nu+\frac{g-2}{2}\delta\alpha_l)}\left(1+\left(\beta_l - \frac{\frac{1}{2}(2-g)\delta\alpha_l - 2}{8(1-\nu)}\right)x^2\right) + \left(\frac{1}{2}x\right)^{2\nu}\mathrm{e}^{-\mathrm{i}\pi\nu}} \tag{60}$$

with $\nu = |m-\delta|$. From this formula it can be seen that $B_m(k)$ vanishes in the limit $R \to 0$, all other parameters being fixed. This is meaningful in the case $g < 2$ where there are no bound states.

It was shown above that in order to have finite bound-state energies for $g > 2$, the limit $R \to 0$ must be taken as $g \to 2$.

For $0 \leqslant \delta < 1$, one must use the substitution (53) which gives

$$B_0(k) \approx \frac{\mathrm{i}\sin\pi\nu}{\mathrm{e}^{-\mathrm{i}\pi\nu} - \frac{g-2}{2}\delta\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\left(\frac{k}{2}\right)^{-2\delta}} \tag{61}$$

i.e. the same formula as in the method of self-adjoint extensions (27).

For a flux larger than one, $\delta = N + \bar{\delta} > 1$, one has to use the substitution (56), which gives

$$B_N(k) = \frac{1}{2}\left(\mathrm{e}^{2\mathrm{i}\pi\bar{\delta}} - 1\right) \qquad (m = N) \tag{62}$$

and

$$B_m(k) \approx R^{2(\delta-m-1)} \to 0 \qquad (m = 0, 1, \ldots, N-1). \tag{63}$$

Therefore, the additional scattering takes place only for the highest angular momentum.

5. Conclusions

We considered a charged quantum mechanical particle with a spin-$\frac{1}{2}$ and gyromagnetic ratio $g \geqslant 2$ in the field of a magnetic string. As is known, the interaction of the charge with the string is the well kown Aharonov–Bohm effect and the contribution of the magnetic moment associated with the spin in the case $g = 2$ yields additional scattering and zero modes (one for each flux quantum). An anomaly of the magnetic moment (i.e. $g > 2$) leads to bound states. We considered two methods for treating the case $g > 2$.

For an ideal string, the interaction of the spin with the magnetic field (5) is point-like and singular; i.e. the magnetic field containes the two-dimensional δ-function (3). A mathematical approach to treating this is the method of self-adjoint extension. It yields a family of operators labelled by a real parameter λ. For all values of this parameter, there is an additional scattering amplitude (24) resulting from the contribution of the magnetic moment; if $\lambda < 0$, there is one bound state (26). The main goal of the extension is to include a singular solution (21) and (25) into the domain of the Hamilton operator (15). It should be remarked that this method–although mathematically correct (or, at least, may be correct)—is not satisfactory from the physical point of view, because the parameter λ of the extension does not correlate to physical parameters like the gyromagnetic ratio (which does not enter this method at all).

A different approach is to consider non-singular flux tubes and the shrinking of the tube radius to zero. This is equivalent to regularizing the δ-function in the magnetic field by some less singular profile. We used three models for which the Schrödinger equation (12) can be solved explicitly (it is actually a Pauli equation in this case). The coventional result is that there is an additional scattering due to the magnetic moment and that there are bound states for $g > 2$. This is not surprising since the existence of zero mode for $g = 2$ suggests the binding of a particle as long as there is an additional attractive force. For g slightly larger than 2, there are $N + 1$ bound states, where N is the integer part of the flux. In general, the dependence of the energy levels κ_m on g and on the flux is complicated; this can be seen in the figure.

We considered the case $g \to 2$. In this case, all κ_m tend to zero. For $m = 0, 1, \ldots, N-1$, the solutions turn to be the above-mentioned zero modes; for $m = N$, the solution vanishes.

The limit $R \to 0$, all other parameters being fixed, is not physical for $g > 2$. The reason is that the bound state energies κ_m enter the defining equation (50) multiplied by the flux tube radius R; this is also implied by general dimensional considerations. Therefore, κ_m tend to infinity for $R \to 0$.

One possibility in treating this problem is to tend the gyromagnetic ratio g in the initial equation (14) to 2 along with $R \to 0$. Although this is not meaningful for real physical particles, it can be viewed as a sort of renormalization of g brought about by the increasing singularity of the potential as $R \to 0$. On the other hand, it allows one to establish the connection with the method of self-adjoint extension. Two cases have to be distinguished. Firstly, when the flux is less than unity, $g - 2$ tend to zero proportionally to $R^{2\delta}$, equation (53). For $g > 2$, there is one bound state, its energy is given by equation (54). This is essentially the same situation as in the method of self-adjoint extension and the parameter λ of the extension can be related to the gyromagnetic ratio, equation (55). Thereby the dependence on the parameters of the models used enters the renormalization (53) of the gyromagnetic ratio only. Such a renormalization is known in the mathematical approach of [8] as well. For the scattering amplitude, there is a contribution in addition to the usual Aharonov–Bohm scattering (equation (61)). It is given by the same formula as in the method of extension, equation (28). Therefore, for $\delta < 1$ both approaches are equivalent.

One observes a different feature for a flux larger than unity. In order to keep all bound state energies finite, one is forced to tend $g-2$ to zero in proportion to R^2 (see equation (56)), i.e. much faster than in the previous case. Thereby the energy of the state κ_N (with the highest angular momentum $m = N$) tends to zero, whereas for $m = 0, 1, \ldots, N - 1$ the energies κ_m are finite. The additional scattering in this case takes place for $m = N$ only (see equation (62)); for $m = 0, 1, \ldots, N - 1$, $B_m(k)$ (equation (63)) vanishes for fixed parameters. This is clear because the wavefunctions in this case are concentrated in the region of small R. For sufficiently high momenta k, scattering can also be expected in this case. We conclude, therefore, that for fluxes larger than one, the two approaches yield different results. This must not be regarded as a contradiction, for the following reasons. In the method of self-adjoint extension, the input information, which is contained in the Hamilton operator (15) makes reference neither to the magnetic moment of the particle, nor to the magnetic field $\mathcal{H}$ (3), nor to the integer part N of the magnetic flux (it enters in the combination $m - \delta$ only). Therefore, we can conclude that the method of self-adjoint extension is only applicable for fluxes less than unity, or that a flux larger than unity is too singular for being described by the method of self-adjoint extension.

The physically interesting case is to keep the flux tube radius small but finite. The natural scale for the smallness of $g - 2$ is given by the anomaly $a_e = (g - 2)/2$ (6) of the magnetic moment of the electron and, since we use a non-relativistic equation, by requiring the radius of the tube not to be too small so as to have non-relativistic bound state energies $\kappa_m \ll m_e$ (m_e is the electron mass). Under these conditions there are approximately $N + 1$

bound states. The exact number is given by equation (49) and depends on the model used. However, this dependence is weak for small a_e as can be seen from equation (49), where α_i enters multiplied by $g-2$. Furthermore, it should be emphasized that the physical restrictions to R and $g-2$ allow the flux tube to be thin in the sense that $\kappa_m R << 1$ is possible; i.e. the orbit size of the bound states is much larger than the radius of the flux tube.

We conclude, therefore, that for a gyromagnetic ratio larger than 2 and for real physical parameters, the flux tube cannot be shrunk to a line. A natural extension of these investigations would be a consideration of the Dirac equation with an additional magnetic moment (i.e. including a term $(g-2)\sigma^{\mu\nu}F_{\mu\nu}$). In that case, the limitation to the bound state energy being non-relativistic is not necessary and smaller R can be considered. Furthermore, one can speculate that the anomaly of the magnetic moment, which is known to decrease in strong magnetic fields [15], will eventually influence the limit $R \to 0$.

A further open question is whether the interaction which comes from the anomaly a_e of the magnetic moment can be treated in perturbation theory with respect to a_e starting from the known solutions (especially from the zero mode of [2]) for an arbitrary profile of the magnetic field inside a finite flux tube.

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